

Stability of inflationary solutions driven by a changing dissipative fluid

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Abstract

In this paper the second Lyapunov method is used to study the stability of the de Sitter phase of cosmic expansion when the source of the gravitational field is a viscous fluid. Different inflationary scenarios related with reheating and decay of mini-blackholes into radiation are investigated using an effective fluid described by time-varying thermodynamical quantities.

1 Introduction

In recent years considerable attention has been paid to the bulk-viscosity driven inflationary scenario. This is only natural since the effect of bulk viscosity in an expanding universe is to reduce the equilibrium pressure. Therefore one may wish to know whether this effect could be strong enough to render a large negative effective pressure that leads to inflation.

Fundamental strings can create an initial cosmological state either of exponential or power law inflation followed by a smooth evolution towards the typical Friedmann decelerated expansion dominated by the stress-energy tensor of a radiation fluid. This connection is natural because the string-driven inflationary expansion may arise due to the spontaneous quantum production of fundamental strings on scales larger than the horizon, [1] and a phenomenological bulk viscosity can be used to describe the effect of particle production [2], [3]. For some cosmological implications of fundamental strings see also [4] and references therein.

Inflationary scenarios have usually been associated with the dynamics of a spatially homogeneous scalar field such that its potential energy overpowered the kinetic energy, and the equation of state the of vacuum $p_\phi = -\rho_\phi$ was satisfied. If at the time of interest the scalar field dominated any other form of energy, then the cosmic scale factor increased exponentially with time [5]. The fact that $\ddot{a}(t) > 0$, where $a(t)$ is the scale factor, has several nice consequences, among others a scale invariant spectrum of initial density perturbations, the one more likely to be compatible with observation [6].

However inflation, either exponential or power-law, can in principle be driven by any mechanism that renders the total hydrostatic pressure negative, such as bulk viscous pressure associated with non-adiabatic expansion in FLRW universes -the effectiveness of this mechanism has been discussed in the literature [7]. Usually in these models a mixture of relativistic and nonrelativistic (heavy) particles is assumed at some early phase of cosmic expansion. There the bulk viscosity can be very large and may drive inflation. After the decay of these heavy particles bulk viscosity vanishes, terminating the inflationary phase, and returning to the radiation dominated Friedmann universe [8].

Very often a viscous pressure represents only a small perturbation to the equilibrium (hydrostatic) pressure of the fluid. However, as is well known -see [9] and [10]- the effect of particle decay can be phenomenologically understood as a dissipative pressure, and this one can be very large depending on how big the decay rate is. This approach was put on a solid footing by Triginer *et al.* [11].

Stability of the de Sitter solution in cosmological models where the source of the metric is a dissipative fluid obeying some causal transport equation has been investigated by several authors. For the truncated version of the transport equation for viscous pressure, stability has been investigated in [8] and [12]. In [13], [14], [15] and [16] the full version of the transport equation -see (5) below- was adopted. The nonlinear full version [17] was used in [18]. The rationale behind causal transport equations can be found in [19], for a short introduction see [20].

Here we introduce a new approach by using the second method of Lyapunov [21] to examine the asymptotic stability of the de Sitter solutions. This provides useful information on the dynamical behavior of the system, not only near the stationary solutions but far away from them as well.

In section 2 the basic equations and Lyapunov's criterion for stability are presented and applied to different situations in which some or other of the three key parameters are held constant and the rest allowed to vary. In

section 3 the whole set of parameters are assumed to vary. Finally section 4 summarizes the main findings of this work.

2 Asymptotic stability

2.1 General setting

The spatially-flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe is described by the metric

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2) \quad (1)$$

Because of the spatial isotropy and homogeneity assumptions, velocity gradients causing shear viscosity and temperature gradients leading to heat transport are absent. The only possible dissipative term corresponds to the bulk viscosity (which as mentioned before may be interpreted as the effect of particle production). So the energy-stress tensor of this viscous fluid is given by

$$T^{ab} = \rho u^a u^b + (p + \pi) h^{ab} \quad (2)$$

where ρ is the total energy density of the cosmic fluid, p its equilibrium pressure, π the dissipative scalar pressure, u^a the four-velocity, normalized so that $u^a u_a = -1$ and the tensor $h^{ab} = g^{ab} + u^a u^b$ projects any tensorial quantity into the hypersurface orthogonal to u^a .

The corresponding Einstein field equations read

$$3H^2 = \kappa\rho \quad (3)$$

$$\dot{H} = -\frac{\kappa}{2}(\rho + p + \pi) \quad (4)$$

where $H \equiv \dot{a}/a$ is the Hubble function, and κ is Einstein's gravitational constant. We use units such that $c = k_B = \hbar = 1$, so that $\kappa = 8\pi/M_P^2$, where M_P is the Planck mass. An over-dot denotes derivative with respect to cosmic time.

As is well known, the dissipative pressure obeys the causal evolution equation [22], [20]

$$\pi \left[1 + \zeta \left(\frac{\tau}{2T\zeta} u^a \right)_{;a} \right] + \tau \dot{\pi} = -3\zeta H \quad (5)$$

where ζ indicates the phenomenological coefficient of bulk viscosity, T the temperature of the cosmic fluid, and τ the relaxation time associated to the dissipative pressure. Usually the latter is given by the kinetic theory of gases or by a fluctuation-dissipation theorem or both [24]. Provided the divergence on the left hand side is small the last equation can be approximated by

$$\pi + \tau \dot{\pi} = -3\zeta H \quad (6)$$

In view of equation (13) -see below- and the frequently made assumption that $\zeta \propto \rho$, which is very natural for radiative-like fluids [23], [24], the above approximation may be justified for expanding regimes with $H^2 \gg \dot{H}$, something to be expected at the commencement of the Universe expansion. In any case the two above equations meet the requirements of causality and stability to be fulfilled by any physically acceptable transport equation [26]. At this point it is expedient to introduce the polytropic index

$$\gamma \equiv 1 + \frac{p}{\rho} \quad (7)$$

which in general depends on time. This is obvious for a cosmic fluid consisting of a mixture of massless and massive particles, as these two components redshift at different rates and consequently γ varies with time.

This set of equations combines to

$$\ddot{H} + 3\gamma H \dot{H} + \tau^{-1} \left[\dot{H} + \frac{3}{2} (\gamma + \tau \dot{\gamma}) H^2 - \frac{3}{2} \zeta H \right] = 0 \quad (8)$$

and the latter can be recast as

$$\frac{d}{dt} \left[\frac{1}{2} \dot{H}^2 + V(H) \right] = D(H, \dot{H}) \quad (9)$$

Here, the left hand side is the time derivative of a Lyapunov function (see the appendix) with

$$V(H) = \frac{\zeta}{2\tau} \left(\frac{\gamma + \tau \dot{\gamma}}{\zeta} H^3 - \frac{3}{2} H^2 \right) \quad (10)$$

and

$$D(H, \dot{H}) = - \left(3\gamma H + \tau^{-1} \right) \dot{H}^2 + \frac{1}{2} \tau^{-2} \left(\tau \dot{\gamma} - \dot{\tau} \gamma + \tau^2 \ddot{\gamma} \right) H^3$$

$$+ \frac{3}{4} \tau^{-2} (\zeta \dot{\tau} - \dot{\zeta} \tau) H^2 \quad (11)$$

Assuming that the ratio ζ/τ is bounded $V(H)$ has two extrema (see Fig. 1), a maximum at $H = 0$, and a minimum at

$$H = \frac{\zeta}{\gamma + \tau \dot{\gamma}} \quad (12)$$

Inserting ζ from (12) in (11) we get

$$D = - \left(3\gamma H + \tau^{-1} \right) \dot{H}^2 - \frac{3}{4} \tau^{-1} (\gamma + \tau \dot{\gamma}) H^2 \dot{H} - \frac{1}{4} \tau^{-2} (\tau \dot{\gamma} - \dot{\tau} \gamma + \tau^2 \ddot{\gamma}) H^3 \quad (13)$$

In the simplest case when the functions γ , τ and ζ are constants, D is seminegative definite in a neighborhood of the de Sitter solution $H_1 = \zeta/(\gamma + \tau \dot{\gamma})$ and the Lyapunov function has a time independent upper bound for large times, implying the stability of this solution [21]. On the other hand, when H is slowly varying we can neglect \dot{H} in (13); thereby D reduces to

$$D \approx - \frac{1}{4\tau^2} (\tau \dot{\gamma} - \dot{\tau} \gamma + \tau^2 \ddot{\gamma}) H^3 \quad (14)$$

Provided D is negative definite, and taking into account that the Lyapunov function has an infinitesimal upper bound in the neighborhood of point $(H_1, 0)$ of the phase space (H, \dot{H}) , the de Sitter solution (12) is asymptotically stable [21].

In contrast with the usual inflationary scenarios driven by a large effective cosmological constant necessarily accompanied by strong supercooling and reheating, dissipative processes in our scheme reheat the medium gently. In the following, several simple cases when either γ or τ remain constant are investigated.

2.2 Variable τ

In the first instance (12) is asymptotically stable when τ is a decreasing function, while in the second case stability occurs when γ increases slowly. Thus, for instance, if γ is a constant and τ is a function that decreases in a first stage and then increases, we have that there is a first period of exponential inflation with $H = \zeta/\gamma$, and then a graceful exit.

For instance, using the relationship for the speed v of the dissipative signal

$$\frac{\zeta}{\tau} = v^2 \gamma \rho \quad (15)$$

derived in [20], we obtain

$$\tau = \frac{1}{3v^2 H} \quad (16)$$

and therefore τ will present a minimum provided v^2 has a maximum. In this case it may be said that the dissipative effect both drives an inflationary stage and then causes the exit from it.

Another way to implement this scenario is as follows. Assume the cosmic fluid is modelled by a mixture of radiation and heavy particles (or massive modes of fundamental strings) that decay at a very high rate into (more stable) lighter particles (less massive modes), with high or moderate multiplicity. Since the relaxation time for the interaction between radiation and massive particles is on very general grounds given by $\tau \approx (n\sigma)^{-1}$, where n denotes the number density of massive particles and σ the interaction cross-section, which can be constant, there will be two stages. The first one corresponds to the decay of the more massive particles; there (despite the expansion) n will augment and τ decrease accordingly. The second stage will commence when most of these particles have decayed; then the expansion will make n decrease and correspondingly τ will increase [25].

2.3 Variable γ and decay of massive particles

This second case will be useful to model the cosmic evolution during the inflationary period by allowing massive (dust) particles to decay into relativistic particles [27]. Thus one may have simultaneously inflation and reheating.

Exit from inflation occurs when particle production ceases and the fast dilution of relativistic particles makes γ decrease again. An interesting model for the reheating-inflationary stage arises when τ and ζ are constant but γ varies in such a way that H_1 remains constant. Using (11) we see that D is semidefinite negative and therefore the de Sitter stage is stable. In this case $\gamma(t)$ takes the form

$$\gamma(t) = \gamma_0 + C_2 e^{-t/\tau} \quad (17)$$

and $H_1 = \zeta/\gamma_0$. It is straightforward to implement a cosmological model in which $\gamma = 1$ at $t = t_1$ and $\gamma = 4/3$ at $t = t_2$, with $t_2 > t_1$

$$\gamma(t) = 1 + \frac{e^{-t/\tau} - e^{-t_1/\tau}}{3(e^{-t_2/\tau} - e^{-t_1/\tau})} \quad (18)$$

The evolution of $\gamma(t)$ depends on the relationship between the timescales τ and $t_2 - t_1$. When $\tau \ll t_2 - t_1$ the relativistic stage is reached very quickly. This corresponds to a universe initially dominated by very massive (dust) particles, that spontaneously decay into radiation at a high initial rate [27]. On the other hand, when $\tau \gg t_2 - t_1$, the growth of $\gamma(t)$ becomes milder.

Since the detailed behavior of a dissipative relativistic fluid in a FLRW universe strongly depends on the thermodynamical properties of the cosmic fluid and generally these are poorly known, we are led to use the Boltzmann gas as the cosmological medium to obtain analytical results. Thus, while a Boltzmann gas is not a realistic model for the cosmological fluid in the actual Universe, it is a fluid for which the thermodynamical properties are well enough established by relativistic kinetic theory to allow us to build up precise models. The thermodynamic properties of the Boltzmann gas may be described by the dimensionless inverse temperature $z = m/T$, the relativistic chemical potential α and a constant $A_0 = m^4 g_*/(2\pi^2)$, where m is the particle mass and g_* the spin weight of the fluid particles. The ideal gas law has the form

$$p = A_0 e^{-\alpha} \frac{K_2(z)}{z^2} \quad (19)$$

$$\rho = A_0 \left[\frac{K_1(z)}{z} + 3 \frac{K_2(z)}{z^2} \right] \quad (20)$$

where K_n are modified Bessel functions of the second kind. Then, inserting (19) and (20) in (7), and assuming a vanishing chemical potential, we obtain

$$\gamma(z) = 1 + \frac{K_2(z)}{zK_1(z) + 3K_2(z)} \quad (21)$$

Equating (18) and (21) we can describe the continuous process of decay of mini-black holes from $t = t_1$ when the black hole energy density dominates the Universe, until t_2 when the black holes have completely evaporated away and the Universe is radiation-dominated. In so doing we are implicitly assuming that all the black holes have the same mass and therefore the same temperature, and that this one equals the temperature of the

massless component of the cosmic fluid at the beginning of the evaporation. Scenarios in which this situation may occur have been reported in the literature -see for instance [28], [29] and [30]. In [28] black holes are formed as consequence of the first order phase transition from the false to the true quantum vacuum; in [29] because of collisions between bubbles of the new phase, and in [30] by quantum fluctuations at the end of hybrid inflation. In all three cases these abundantly produced mini-black holes dominate the Universe and their subsequent explosive evaporation into lighter particles can be modeled as a dissipative pressure.

Following [31] the equation for the evolution of the temperature of the radiation fluid is

$$\frac{\dot{T}}{T} = \frac{9H^2\zeta}{T\partial\rho/\partial T} - 3H\frac{\partial p/\partial T}{\partial\rho/\partial T} \quad (22)$$

Inserting (19), (20) and (7) in (22) we obtain

$$\frac{z'}{z} = \frac{12K_2(z) + 3zK_1(z) - Bz^2}{12K_2(z) + 5zK_1(z) + z^2K_0(z)} \quad (23)$$

where $B = 9H\zeta/A_0$ and $' \equiv d/Hdt$. From this equation it is easy to see that z' is negative for large z , which is in accord with the reheating scenario of above (see Fig. 2).

We can explicitly obtain the time dependence of this temperature near t_1 and t_2 . Expanding (18) about $t = t_1$ and (21) for $z \rightarrow \infty$, we obtain $T \propto t - t_1$. In the opposite limit (i.e. $t \rightarrow t_2$ and $z \rightarrow 0$) it follows that $T \propto (t_2 - t)^{-1/2}$. Thus -as found in similar scenarios [32]- the production of relativistic particles at the final stage of the black holes evaporation is accompanied by a huge increase of the temperature of the cosmic fluid. As for the temperature of the black hole component we must say that this is essentially zero since the black holes behave as a dust fluid (only that its “particles” emit radiation and (simultaneously) absorb the ambience fluid). Nevertheless, one may choose to ascribe a temperature to each individual mini-black hole of mass M by the Hawking relationship $T_{bh} \propto M^{-1}$ -see [33]. The evolution of this temperature is governed by the sum of two terms. One of them comes from the black hole evaporation $\propto M^{-4}$ [33], and the other one comes from the accretion. The latter term is more complicated and not of much interest for our purposes here. In any case, the fate of the black holes (assuming they do not leave any stable relic behind) is their complete disappearance by yielding their whole mass to the radiation fluid.

The entropy production per unit volume in the latter is given by [20]

$$\dot{S} = \frac{\pi^2}{\zeta T} = \frac{9\gamma^2\zeta^3}{\gamma_0^4 T} \quad (24)$$

It has the limiting behavior

$$\dot{S} \simeq \frac{3^7 \tau (1 - e^{-x})^5}{m (3e^{-x} - 2)^4 (t - t_1)}, \quad t \rightarrow t_1 \quad (25)$$

$$\dot{S} \simeq \frac{6^{9/2} \zeta^3 (e^{-x} - 1)^4 \sqrt{t_2 - t}}{m \sqrt{\tau} (3e^{-x} - 2)^4 \sqrt{e^x - 1}}, \quad t \rightarrow t_2 \quad (26)$$

where $x = (t_2 - t_1)/\tau$. Thus we see that the entropy production rate is very high at the beginning of the evaporation, while it decreases sharply at the final stage of this process. At first sight this may seem counter-intuitive if one has in mind that the final stage of black hole evaporation is explosive (when one adheres to the Hawking picture as we do). However, our assumption of $\zeta = \text{constant}$ implies that the aforementioned accretion renders the evaporation rate much milder. The net rate of radiation particle production per mini-black hole and unit of volume roughly varies as $(\rho + p)^{-1}$, where in this case ρ and p refer to the radiation fluid only [34].

We note that the opposite process, i.e. the one in which the accretion of radiation by the black holes overpowers the evaporation of the latter, and as a consequence the radiation is entirely eaten up by the black holes, is ruled out by the second law of thermodynamics [32].

2.4 Variable γ and τ

Assuming now that both γ and τ change in time, with a time scale for γ much larger than τ , we may neglect the term $\tau^2 \ddot{\gamma}$ in equation (14), whence we are left with $D = -(1/4) (\gamma/\tau) \cdot H^3$. Thus, the exponential inflation is stable provided γ/τ is an increasing function, and whenever it begins to decrease, the de Sitter stage ends.

The solution $H = 0$ corresponds to the Minkowski solution. To analyse its stability we linearize equation (8) about it,

$$\ddot{H} + \tau^{-1} \dot{H} - \frac{3}{2} \zeta \tau^{-1} H = 0 \quad (27)$$

The roots of the characteristic polynomial are

$$\lambda_{\pm} = \frac{1}{2\tau} \left(-1 \pm \sqrt{1 + 6\zeta\tau} \right) \quad (28)$$

Since ζ and τ are positive definite quantities it follows $\lambda_- < 0 < \lambda_+$ and therefore the Minkowski solution is unstable. There exists, however, a one-parameter family of solutions that approaches a flat spacetime solution at large times. There is in addition a one-parameter family of solutions that starts from a Minkowski spacetime in the far past and evolves towards a stable de Sitter solution. The time-reversal of the latter is nonsingular and corresponds to a spatially flat universe.

3 General case

In this section we consider the de Sitter solution (12) and assume that γ, τ and ζ are arbitrary functions. This more general situation may occur during the decay of massive particles into lighter ones and also during the decay of four-dimensional fundamental strings into massive and massless particles -admittedly this second possibility is more speculative. The differential equation to solve is

$$\gamma + \tau\dot{\gamma} = \frac{\zeta}{H_1} \quad (29)$$

We introduce a new dimensionless independent variable $d\eta = dt/\tau$. The reason for using a dimensionless equation is that the equilibrium point of the differential equation (29), describing exponential inflationary models, will represent self-similar cosmological models. Then the general solution reads

$$\gamma = \frac{1}{H_1} e^{-\eta} \left(\int d\eta \zeta e^{\eta} + C \right) \quad (30)$$

where C is an arbitrary integration constant. We will use this result to present two simple models. First we consider that $\zeta = \zeta_0$, a constant. Thereby

$$\gamma(\eta) = \frac{1}{H_1} (\zeta_0 + C e^{-\eta}) \quad (31)$$

A natural choice is $\eta = \nu \ln t$, ν being a positive constant, that represents a linear dependence between τ and the cosmological time t . Then we obtain

$$\gamma(t) = \gamma_0 \left(1 + \frac{\bar{C}}{t^\nu} \right) \quad (32)$$

with $\gamma_0 = \zeta_0/H_1$. This expression is monotonic increasing (decreasing) for $\bar{C} < 0$ ($\bar{C} > 0$). For the case that $\bar{C} = (t_1 t_2)^\nu / (4t_2^\nu - 3t_1^\nu)$ and $\gamma_0 = (4t_2^\nu - 3t_1^\nu) / (3(t_2^\nu - t_1^\nu))$, this model describes decay of massive particles into radiation beginning at t_1 and finishing at t_2 .

Now inserting (32) into (15) we obtain an expression for the dissipative contribution to the speed of sound

$$v^2 = \frac{\nu}{3H_1} \frac{t^{\nu-1}}{t^\nu + \bar{C}} \quad (33)$$

This is a monotonic decreasing function that in the limit $t \rightarrow \infty$ behaves as $v^2 \sim \nu/(3H_1 t)$. Thus we require that $v(t_1) \leq 1$.

Considering now that $\zeta = \zeta_0 e^{-\eta}$, the expression (30) gives

$$\gamma(\eta) = \frac{\zeta}{H_1} (\eta + C) \quad (34)$$

In order to simplify the calculations we restrict to constant v ; in this case we obtain

$$\Delta t = \frac{1}{3H_1 v^2} \ln |\eta + C| \quad (35)$$

In order to have $\Delta t > 0$, we choose $\eta + C > 0$; then

$$\gamma(\Delta t) = \gamma_0 \exp \left(C + 3H_1 v^2 \Delta t - e^{3H_1 v^2 \Delta t} \right) \quad (36)$$

This expression has a maximum at $\Delta t = 0$, and assuming that $\gamma(0) = 4/3$ we find that there is a phase of decay into radiation starting when $\gamma = 1$ at $3H_1 v^2 \Delta t = -0.8678$ that reaches a relativistic gas state at $\Delta t = 0$. After that there is a condensation phase back into nonrelativistic matter that ends at $3H_1 v^2 \Delta t = 0.6736$ when γ returns to 1. A scenario compatible with the latter phase is the quantum tunneling of radiation into black holes [35]. This may arise very naturally because of the instability of the hot radiation against spontaneous condensation [36]. (It is altogether different from the whole disappearance of the radiation by black hole accretion).

During this period both the viscosity coefficient

$$\zeta(t) = \zeta_0 \exp \left(C - e^{3H_1 v^2 \Delta t} \right) \quad (37)$$

and the relaxation time

$$\tau(t) = \frac{1}{3H_1 v^2} e^{-3H_1 v^2 \Delta t} \quad (38)$$

are monotonic decreasing functions.

Again we may interpret this behavior in terms of a two-fluid model, where the viscosity coefficient arises because of the particle production process from the decay of massive nonrelativistic particles into light ones. Shortly after the beginning of the decay the particle production rate is large and the energy density of the fluid becomes dominated by the light component. Later on, as the decay rate slows down the effect of adiabatic dilution by the fast exponential expansion of the universe turns out to be more important. Therefore the nonrelativistic particles dominate again, since their number density goes down as a^{-3} , while the relativistic component goes down at the faster rate of a^{-4} .

4 Conclusions

With the help of the Lyapunov method we have studied the stability of cosmic inflationary expansions driven by a dissipative fluid whose transport equation is of causal type. We required a slow transition from the symmetric to the broken phase. This transition may be even quasistatic. Exponential inflation occurs during the transition whenever the ratio $\zeta / (\gamma + \tau \dot{\gamma})$ remains constant, or at most varies slowly in time. The behavior of the scale factor changes gently from exponential expansion to Friedmannian ($a(t) \propto t^n$, with $0 < n < 1$) once the heavy particles have either decayed or become sufficiently diluted, in this way rendering the viscosity negligible.

We have found that the de Sitter solution is asymptotically stable for a wide set of reasonable fluid quantities of the dissipative cosmological medium. Thus the condition for inflation appears natural, and the extremely fast supercooling followed by an intense reheating, so frequent in the literature, is avoided.

To obtain analytic results for the continuous decay processes of decay of mini-black holes we have modeled the fluid by a Boltzmann gas. Thus we are able to obtain the time evolution of the temperature from non-relativistic to the ultrarelativistic regime reflecting in an increase of the adiabatic index from 1 to 4/3. In this way a reheating phase is shown to occur simultaneously with exponential inflation.

Concerning the structural stability of the de Sitter solution relative to the spatial geometry, we have neglected the effect of curvature in comparison with the energy density and effective pressure during an exponential inflationary stage. In the case of non-causal bulk viscosity, where π is algebraically determined by H , it is relatively straightforward to investigate this question [37, 2, 3]. By contrast, in the causal theory π is no longer algebraically determined by H but satisfies a transport equation that couples it differentially to the expansion. Curvature introduces the scale factor explicitly into the Friedmann equation, and makes it much harder to decouple the equations. Thus the effect of curvature in the causal case is far more difficult to determine in general. This will be the subject of future work.

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Appendix

The stability of a solution of equation (8) can also be studied by linearization about this solution. However, this procedure leads to a linear differential equation for the perturbation with time-dependent coefficients, through three unspecified functions: $\zeta(t)$, $\gamma(t)$ and $\tau(t)$. Consequently it is extremely difficult to determine the behavior of the perturbation for large times, as it requires to calculate the perturbation as a functional of these coefficients. This is why we have chosen the second method of Lyapunov. This method has led us to some general qualitative results about the stability of the de Sitter solution.

For a time-dependent Lyapunov function we have used the following theorems [38]:

1. *If a function V exists which is defined and whose derivatives V' is a semidefinite function whose sign is contrary of that of V , then the solution $x = 0$ of*

$$x'_i = f_i(t, x_1, \dots, x_n), \quad i = 1, 2, \dots, n, \quad (A1)$$

is stable.

2. If a function V exists which is definite, and has an infinitesimal upper bound, if the derivative V' is also a definite function whose sign is contrary to that of V , then the solution $x = 0$ of (A1) is asymptotically stable.

In our case $x \equiv (H - H_1, \dot{H})$ and the Lyapunov function is

$$\frac{1}{2}\dot{H}^2 + \frac{\zeta}{2\tau} \left(\frac{\gamma + \tau\dot{\gamma}}{\zeta} H^3 - \frac{3}{2} H^2 \right)$$

which clearly satisfies the hypothesis of the theorem provided that ζ/τ is bounded (see equation (15)), in a neighbourhood of the de Sitter solution $H_1 = \zeta/(\gamma + \tau\dot{\gamma})$.

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Figure Captions

Figure 1. Potential $V(H)$ of the equivalent mechanical system defined by equation (10).

Figure 2. Plot of the dimensionless time derivative of the inverse temperature z' for $B = 0.1$.



